

Sparse Domination and the A_2 Conjecture

Extended Abstract

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Introduction

Historical Context

The study of weighted inequalities in Harmonic Analysis as we know them today started in the 1970's, with the definition of the so-called A_p weights¹ (also sometimes referred to in the literature as Muckenhoupt weights²). For $1 < p < \infty$, these weights ω are exactly the L^1_{loc} non-negative functions for which the maximal function is a bounded operator $L^p(\omega) \rightarrow L^p(\omega)$ and a bit of work shows that an equivalent characterization is that

$$[\omega]_p := \sup_{\text{cubes } Q} \left(\frac{1}{|Q|} \int_Q \omega \right) \left(\frac{1}{|Q|} \int_Q \omega^{1-p'} \right)^{p-1}$$

is finite. This value $[\omega]_p$ is called the A_p characteristic of ω and it is in a way a measure of its size.

A lot has been known about weighted inequalities for A_p weights since their original study³:

- Calderón-Zygmund operators satisfy strong weighted L^p bounds when the weight is in A_p ($1 < p < \infty$), due to R. Coifman and C. Fefferman in [CF74];
- In the particular case of the Hilbert transform, ω being in A_p is necessary in order for the Hilbert transform to satisfy strong weighted bounds, as in the case for the maximal function (see [HMW73]);
- In 1984, Rubio de Francia proved an extrapolation theorem (see [Rub84]), which states that proving weighted bounds for any $\omega \in A_p$ immediately proves weighted bounds for weights in any other A_p class $A_{p'}$. Moreover, this extrapolation can be made quantitative, that is, in a way where we keep track of the dependence on the A_p characteristic of the weight.

In more recent years, there was a growing interest in quantitative versions (that is, with explicit dependence on the A_p characteristic $[\omega]_p$) of the strong weighted bounds above. Originally studied in [Buc93], these quantitative versions have proved to be helpful in the context of PDE, for example in [AIS01],

¹It is worth noting that the A_p condition that follows had been previously studied by M. Rosenblum in [Ros62], even if in a different form.

²The characterization of A_1 weights, which we do not focus on for now, is due to B. Muckenhoupt ([Muc72]), who developed a good amount of the classical theory on A_p weights.

³These references come from [Duo01] and [Per19], where one can find more information and references on the history of weighted inequalities.

where a linear dependence of the Beurling-Ahlfors operator weighted bound on the A_p characteristic is crucial to show sharp regularity results⁴.

Through the years, results for specific operators have been proved, a survey of which can be found in sources like the introduction of [Hyt12] or [Ler13a]. In the end, as seen in the timeline in [Ler13a], the ideas in these partial proofs culminated in a proof of the A_2 Conjecture - now A_2 Theorem - by Hytönen in his paper [Hyt12]. Interestingly, the story does not end there: a couple of years later, there were enough approaches to the A_2 theorem that Hytönen published a survey [Hyt14] going over all the theory studied and created in view of a proof of the A_2 conjecture, summed up in the following diagram:

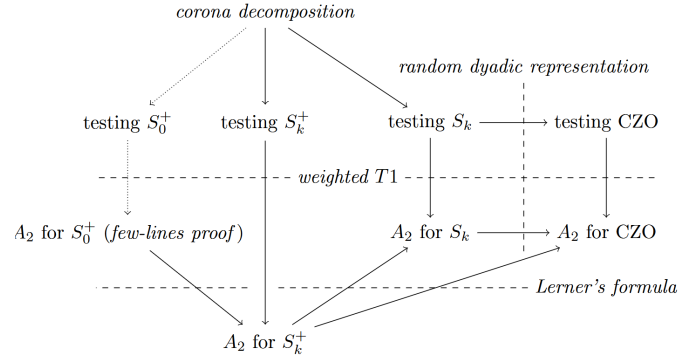


Figure 1: Approaches to the A_2 theorem by 2012, published in 2014 by Hytönen in [Hyt14]

Up to this point, the proof of the A_2 theorem was, even with all the simplifications, quite technical, and relied in a Haar Shift operator decomposition technique originally used to prove the result for the Hilbert transform (by S. Petermichl in [Pet07]) and then generalized to an arbitrary Calderón-Zygmund operator. However, a quick succession of papers gradually simplified the proof:

- Lerner proves that Calderón-Zygmund operators can be bounded by sparse forms in [Ler13b] (in a general Banach norm sense), which can be used to prove a good number of results, notably yielding a simplified proof of the A_2 conjecture ([Moe12]). However, this bound still relied on the representation of the operator in term of the Haar shift operators.
- In Lerner's "A Simple Proof of the A_2 Conjecture" ([Ler13a]), the role of the Haar Shift operator is replaced by the "local mean oscillation decomposition", a first step towards a more elementary proof.
- In [CR16], a duality condition crucial in [Ler13b] is lifted by a sparse bound (of dyadic shifts), which remarkably is a pointwise bound instead of a bound in norm. Such pointwise bounds were independently established in [LN19] as well.
- In Lacey's "An elementary proof of the A_2 bound" ([Lac17]), inspired by the pointwise bounds, the mean oscillation inequality used in [Ler13a] is replaced by exploiting the behavior of dyadic grids, as well as weak bounds for the maximal truncations of the operators.

⁴Both these references are cited in [CMP12], whose introduction also contains pieces of the historical context.

- Finally, in [Ler16] Lerner modified Lacey's proof from [Lac17] into a shorter and elementary proof of A_2 via pointwise sparse domination. This proof, with some of the improvements from the later paper [LO20] by Lerner and Ombrosi ("Some remarks on the pointwise sparse domination"), is arguably the simplest proof of the A_2 conjecture to date, and will be the main focus of this thesis.

Even more remarkable than the body of work produced in search of a proof of the A_2 conjecture, it turns out that the most elementary proof via Sparse Domination is extremely powerful and versatile. In the following years, varied types of Sparse Domination were used to prove weighted inequalities for a very diverse collection of operators, in what Pereyra coined in [Per19] as the Sparse Revolution. In this document, Pereyra surveys the role of dyadic harmonic analysis in the proof of weighted inequalities, detailing the path outlined above and referencing some of the later work it inspired.

Outline

In the first chapter of the thesis, we present the classical theory of A_p weights, including some of the important results referenced above. We lay the groundwork for the work done with A_p weights in the following chapters, as well as highlight the challenges of attempting to apply classical methods to keep track of the dependence on the A_p characteristic in weighted inequalities.

In the second chapter, we begin by introducing the notion of sparse families and forms, and we prove strong weighted results for these operators, with explicit dependence on the A_p constant. We then prove that the maximal function is pointwise bounded by sparse forms, to showcase the main ideas of the proof of the A_2 conjecture in a simpler case, so that the technical details do not obscure the essence of the proof. Finally, we prove the A_2 theorem, in a very similar way to Lerner's proof from [Ler16].

In the third chapter, we show some of the versatility of Sparse Domination, both via the generalized pointwise theorem in [LO20] and the very fruitful bilinear formulation of Sparse Bounds. We also address endpoint and weak bounds and discuss a negative result for multiparameter Sparse Domination.

In this extended abstract, we keep this general structure, giving an overview of the highlights of each chapter.

1 A_p Weights

Before introducing A_p weights, we begin by defining weights and weighted integrals:

Definition 1.1: A weight ω is a non-negative measurable function. The ω -weighted integral of f is defined as $\int f\omega := \int f(x)\omega(x)dx$.

One might be interested in classifying which weights are good to work with, in the context of Harmonic Analysis. It is a reasonable request that for any such weight the Hardy-Littlewood maximal operator, a relatively simple yet extremely important operator in Harmonic Analysis, has "the correct bounds". For $1 < p < \infty$, a clever manipulation of these bounds lead to the standard definition of A_p weights:

Definition 1.2 (A_p weight): A weight ω is called an A_p weight, for $1 < p < \infty$, if it satisfies

$$[\omega]_p := \sup_Q \left(\frac{1}{|Q|} \int_Q \omega \right) \left(\frac{1}{|Q|} \int_Q \omega^{1-p'} \right)^{p-1} < \infty, \text{ where } \frac{1}{p} + \frac{1}{p'} = 1.$$

The quantity $[\omega]_p$ is referred to as the A_p weight characteristic of ω .

It turns out that these are *exactly* the weights that behave well in regards with the maximal operator:

Proposition 1.3: Let ω be a weight. Then, the maximal operator is bounded $L^p(\omega) \rightarrow L^p(\omega)$ if and only if ω is an A_p weight.

These weights ω have a few nice properties⁵:

- They are non-zero almost everywhere in the Lebesgue measure and locally integrable;
- We can control the ω -size of dilations, which is very helpful for technical arguments;
- We have $\omega \in A_p$ iff $\omega^{1-p'} \in A_{p'}$, allowing for duality arguments;
- We have that if $\omega \in A_p$ then it is in every A_q with $q \geq p$.
- Surprisingly, we have that if $\omega \in A_p$ then there is some $q < p$ such that $\omega \in A_q$ ⁶. As a result, $A_p = \cup_{q < p} A_q$.

The theorem of biggest interest for our purposes is the strong $L^p(\omega)$ bound for Calderón-Zygmund operators, which we state with a sketch of proof that hopefully hints to the difficulty of keeping track of the dependence on the A_p characteristic:

Proposition 1.4: Let T be a Calderón-Zygmund operator and $\omega \in A_p$. Then,

$$\|Tf\|_{L^p(\omega)} \lesssim \|f\|_{L^p(\omega)}, \quad 1 < p < \infty.$$

Sketch of Proof: Let $\omega \in A_p$. Then, by the last property above, we have that $\omega \in A_{\frac{p}{s}}$ for some s (take $s = \frac{p}{q}$, where q is given by the last bulletpoint above). Assuming without loss of generality that f is a bounded function of compact support, we have the following chain of inequalities:⁷

$$\int_{\mathbb{R}^n} |Tf|^p \omega \leq \int_{\mathbb{R}^n} M_d(Tf)^p \omega \lesssim \int_{\mathbb{R}^n} M^\#(Tf)^p \omega \stackrel{(\star)}{\lesssim} \int_{\mathbb{R}^n} M(|f|^s)^{\frac{p}{s}} \omega \lesssim \int_{\mathbb{R}^n} |f|^p \omega, \quad \text{q.e.d.} \quad \square$$

The full proof and a general textbook presentation of A_p weights can be found in [Duo01] (chapter 7, theorem 7.11) or in [Gra14]. The main thing to notice in the chain of inequalities is that in the crucial step (★) we introduce this s , which we know exists, but do not know how to construct. From there, the constants depend on s , which depends on ω in a very mysterious way, and therefore all hope to keep track of the dependence on ω is lost at this step. Indeed, the proof we present of the A_2 theorem via Sparse Domination uses radically different ideas.

⁵With the exception of the last property, all these hold for $p = 1$, though we do not address the class A_1 in this document.

⁶This is a consequence of the Reverse Hölder Inequality, one of the most important properties of A_p weights. Unfortunately, the proof of this result is not constructive, and it does not give us any information about what q is, in terms of p, ω .

⁷Here, $M^\#(f) := \sup_Q \frac{1}{|Q|} \int_Q (f - f_Q)$ (where $f_Q = |Q|^{-1} \int_Q f$) is the mean oscillation of f , M is the Hardy-Littlewood maximal operator and M_d is the dyadic maximal operator.

2 The A_2 Conjecture and Pointwise Sparse Domination

2.1 Sparse Families of Sets

The main idea in Sparse Domination is to use dyadic calculus to exploit the nice properties of dyadic grids, which allow us to easily write sets of interest as disjoint unions of well-behaved sets. However, it is sometimes the case that disjointness is too strong of a requirement, and it turns out the right notion to work with instead is that of sparseness:

Definition 2.1 (Sparse Collection): A family of sets $\mathcal{S} = \{S_\alpha, \alpha \in A\}$ is η -Sparse if for every α there is some set $E(S_\alpha) \subseteq S_\alpha$ such that $|E(S_\alpha)| \geq \eta|S_\alpha|$ and the $E(S_\alpha)$ are pairwise disjoint.

We say that a family of sets is Sparse if it is η -Sparse, for some $0 < \eta \leq 1$.

We sometimes refer to $E(S_\alpha)$ as the E -set of S_α .

It is clear from the definition above that any pairwise disjoint family is sparse, with $\eta = 1$ and $E(S_\alpha) = S_\alpha$. We now show a couple of examples of more interesting sparse families:

Example 2.2: Let $\mathcal{S} = \{[0, 2^n[, n \in \mathbb{N}^+\}$. This is a $\frac{1}{2}$ sparse family, with $E([0, 2^n[) = [2^{n-1}, 2^n[$. It is clear that $|E([0, 2^n[)| = 2^n - 2^{n-1} = 2^{n-1} = \frac{1}{2}|[0, 2^n[|$ and that the sets in $\{E([0, 2^n[), n \in \mathbb{N}^+\} = \{[2^{n-1}, 2^n[, n \in \mathbb{N}^+\}$ are pairwise disjoint.

Example 2.3: Given a dyadic grid $\mathcal{D}$, any collection of dyadic cubes $\mathcal{S}$ with bounded generation is M^{-1} -sparse, with M the number of generations.

Indeed, let k be the minimal generation in $\mathcal{S}$. Divide each cube in $\mathcal{D}_k$ into M disjoint subsets of equal size and assign each one to a generation (see figure 2). Then, for each cube Q in $\mathcal{S}$ let $E(Q)$ be the union of these chosen subsets which are contained in Q . Then $|E(Q)| = M^{-1}|Q|$ and the $E(Q)$ are disjoint by construction, since two distinct cubes of the same generation are disjoint and given two cubes of different generations, the intersections of their $E(Q)$ with each cube in $\mathcal{D}_k$ will be disjoint as different generations select different subsets.

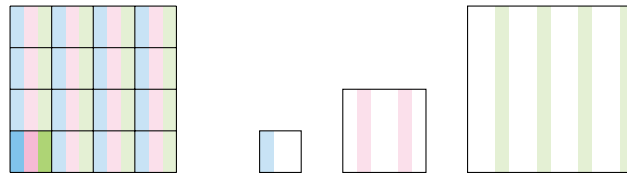


Figure 2: Division of $\mathcal{D}_1$ into 3 disjoint sets of equal size (blue, pink and green), followed by $E(Q)$ for a cube Q of generation 1 in blue, $E(Q)$ for a cube of generation 2 in pink and $E(Q)$ for a cube of generation 3 in green.

2.2 Sparse Forms

The Sparse Domination technique consists on bounding varied operators by Sparse Forms,

$$\mathcal{A}_S f(x) := \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \chi_Q(x), \quad \text{where } \langle f \rangle_Q = \frac{1}{|Q|} \int_Q |f|.$$

with $\mathcal{S}$ a sparse family, in order to prove certain boundedness properties of our operators of interest.

These operators are quite simple, as they are just (infinite) linear combinations of characteristic functions, indexed in a sparse family of sets. The reason why they are so helpful to prove sparse bounds is that not only do they have the strong weighted bounds we intend to prove, but we can also prove an explicit dependence on the A_p constant, which turns out to be sharp for many of the operators we are interested in bounding:

Proposition 2.4: For every sparse family $\mathcal{S}$, every $\omega \in A_p$ and every $f \in L^p(\omega)$, we have the bound

$$\|\mathcal{A}_{\mathcal{S}}f\|_{L^p(\omega)} \lesssim [\omega]_p^{\max(1, \frac{1}{p-1})} \|f\|_{L^p(\omega)}.$$

The proof of this proposition can be found in the main text. Essentially, the fact that we have (infinite) linear combinations of characteristic functions allows us to use duality and introduce integrals over these Q of our test functions. Then, a clever use of bounds for the weighted maximal function and the Hölder inequality makes integrals of ω over the cubes Q pop up, which - almost by sorcery - can be rearranged into exactly the A_p characteristic. That is, the dependence on the A_p characteristic that we so desire can be found using truly elementary methods, which is the big advantage of Sparse Forms.

2.3 Pointwise Sparse Domination of the Maximal Function

The proof of pointwise Sparse Domination works especially well for the dyadic maximal function, due to its interplay with the Calderón-Zygmund decomposition. It is possible to show that such bounds for a dyadic maximal function extend to bounds for the Hardy-Littlewood maximal operator, and a proof can be found in the main text.

So, we fix a one-quadrant dyadic grid $\mathcal{D}^8$. We consider a function of compact support f , and take a dyadic cube Q_0 which contains its support - this is always possible in one-quadrant grids.

The idea of the proof is as follows: for any point $x \in Q'$ such that $M_{\mathcal{D}}f(x) \leq C\langle f \rangle_{Q'}$ (we denote these Q' -good points), $M_{\mathcal{D}}f(x)$ will also be bounded by $C\sum_{Q \in \mathcal{S}} \langle f \rangle_Q$ for any sparse family such that $Q' \in \mathcal{S}$, since all summands are non-negative. So, we want to iteratively add cubes Q_α to $\mathcal{S}$ such that the union of the Q_α -good points over all alphas covers the space (so that every point is bounded by the corresponding $C\langle f \rangle_{Q_\alpha}$ and therefore the sparse form), while ensuring $\mathcal{S}$ is sparse.

For points y outside of Q_0 , it can be checked that $M_{\mathcal{D}}f(y)$ will be exactly the average of f over one of the ancestors of Q_0 , the idea being that they are the smallest dyadic cubes that contain y while also intersecting (in fact, containing, due to the fact that dyadic cubes are either disjoint or contained in each other) Q_0 , and therefore the support of f (see figure 3). So, adding these to the family takes care of Q_0^c and they are sparse by a similar reasoning to Example 2.2.

⁸One-quadrant means that if we consider a dyadic cube and look at their ancestors, they will eventually cover the full space. This is not the case of the standard dyadic grid, which has 2^n quadrants; for example in two dimensions the ancestors of $[0, 1]^2$ cover only the first quadrant.

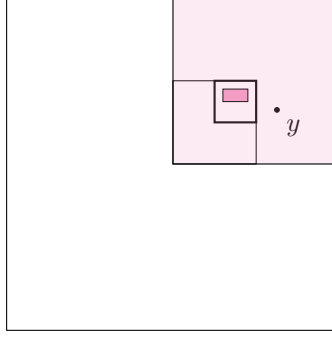


Figure 3: A section of one-quadrant dyadic grid. In dark pink, the support of f , in thick black Q_0 and in thin black its ancestors. For the point y outside, the smallest dyadic square containing y and intersecting the support of f is indicated in light pink.

For points in x we iterate as follows: we can bound the size of the “bad points” of Q_0 using the weak $(1, 1)$ inequality for the maximal function and noting that $\|f\|_1 = \int_{Q_0} f$, since Q_0 was chosen to contain the support of f :

$$|\{x \in Q_0 : M_{\mathcal{D}}f(x) > C\langle f \rangle_{Q_0}\}| \leq C_{M_{\mathcal{D}}} \frac{\int_{Q_0} f}{C\langle f \rangle_{Q_0}} = \frac{C_{M_{\mathcal{D}}}|Q_0|}{C}.$$

That means we can choose $C = 2C_{M_{\mathcal{D}}}$ such that the “bad points” occupy less than half of Q_0 , and as a result the “good points” (which will be our $E(Q_0)$) occupy more than half. Now, the key point is that the Calderón-Zygmund decomposition allows us to write this bad set as a union of cubes, and for each of these cubes we can use a similar reasoning⁹ to extract a good set, then break the bad set into cubes and perform the same operation in each of them, and just keep going until our good sets cover Q_0 almost everywhere¹⁰ (see picture 4).

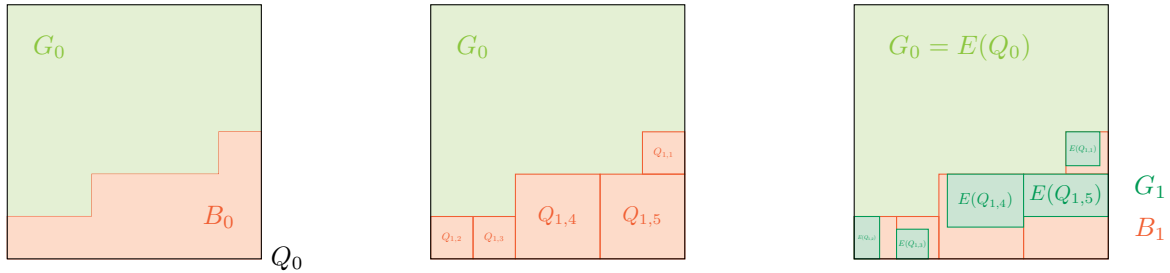


Figure 4: Illustration of the first two steps. In light green G_0 as defined above, the set of “good points”. The “bad set” B_1 in orange is then written as a disjoint union of dyadic squares. For each of which, we define $E(Q_{1,i})$ consisting of the “good points” in that square. We then define G_1 as the union of these new “good points”, and $B_1 = B_0 \setminus G_1$. In these steps, we added Q_0 and $Q_{1,1}, \dots, Q_{1,5}$ to $\mathcal{S}$, with the corresponding E -sets represented in green. Note that these need not be rectangles and will usually in fact be more complicated sets.

⁹A slightly technical argument is needed to justify we can apply our weak inequality to $f\chi_{Q'}$ (with Q' the new cube) instead of f so that the integrals still cancel out, but the reasoning otherwise applies exactly to the next iterations.

¹⁰This is easy to ensure happens in the limit, since we are halving the measure of the bad points at each step of the iteration.

At the end of this process, we have (almost) every point in Q_0 in the union of the good sets, and we have that the cubes are a sparse family, since they were constructed to be disjoint from the cubes in the same iteration step and disjoint from the good points in the previous iteration steps. So, the collection of the cubes will be the sparse family we were looking for, meaning that we have established a pointwise Sparse Domination result for the (dyadic) maximal function.

2.4 The A_2 Conjecture

Recall that the A_2 Conjecture states that the $L^2(\omega)$ weighted bound for Calderón-Zygmund operators depends linearly on $[\omega]_2$:

$$\|Tf\|_{L^2(\omega)} \lesssim [\omega]_2 \|f\|_{L^2(\omega)}.$$

By proposition 2.4, it is enough to prove the following proposition:

Proposition 2.5: Let T be a Calderón-Zygmund operator and f be a compactly supported L^1 function. Then, there is a η/α^n -sparse family $\mathcal{S} = \mathcal{S}(f)$ such that

$$|Tf| \leq C \sum_{\tilde{Q} \in \mathcal{S}} \langle f \rangle_{\tilde{Q}, p} \chi_{\tilde{Q}} \quad \text{a.e.}$$

Proving Sparse Domination for Calderón-Zygmund operators follows the same guiding principles present in the domination of the dyadic maximal function, but needs technical changes in the several points where the specific properties of the dyadic maximal functions were exploited. Here, we give a very macroscopic view of these differences, and a full proof can be found in the main text.

The first difference is in reducing to the local case. For a general Calderón-Zygmund operator there is no canonical choice of cubes that automatically takes care of the complement of a cube containing the support, as is the case of the dyadic maximal function. Although we are still able to restrict our attention to the local case, it comes at a cost, since we need to introduce averages over dilations of dyadic cubes:

Lemma 2.6: Let T be an operator and f be a compactly supported L^1 function such that for every cube Q there is a constant $C > 0$ and a η -sparse family $\mathcal{S}_Q$ contained in Q such that

$$|Tf \chi_{\alpha Q}| \leq C \sum_{\tilde{Q} \in \mathcal{S}_Q} \langle f \rangle_{\alpha \tilde{Q}} \chi_{\tilde{Q}},$$

with $\alpha \geq 3$ and $\alpha \tilde{Q}$ the dilation of the cube $\tilde{Q}$ by α , fixing the centre. Then, we have that there is a η/α^n -sparse family $\mathcal{S}$ such that

$$|Tf| \leq C \sum_{\tilde{Q} \in \mathcal{S}} \langle f \rangle_{\tilde{Q}} \chi_{\tilde{Q}}.$$

The second and main issue comes in our iterative step. Let's say we define a "good set" G and a "bad set" B , and find a way to decompose B into cubes $\{Q_j\}_j$, as to continue the iteration argument in

each of them. Then, we have:

$$|T(f\chi_{\alpha Q})|_{\chi_Q} = |T(f\chi_{\alpha Q})|_{\chi_G} + |T(f\chi_{\alpha Q})|_{\chi_B} = |T(f\chi_{\alpha Q})|_{\chi_G} + \sum_j |T(f\chi_{\alpha Q})|_{\chi_{Q_j}}.$$

In order to iterate in this expression, we would need to have the term $\sum_j |T(f\chi_{\alpha Q_j})|_{\chi_{Q_j}}$ instead of $\sum_j |T(f\chi_{\alpha Q})|_{\chi_{Q_j}}$. That means that we have to control the difference between these two expressions, which is bounded by $\sum_j (|T(f\chi_{3Q})| - |T(f\chi_{3Q_j})|)_{\chi_{Q_j}} \leq \sum_j |T(f\chi_{3Q \setminus 3Q_j})|_{\chi_{Q_j}}$. In order to do that, we define the operator

$$\mathcal{M}_{Q_0, T}f(x) := \sup_{\tilde{Q} \subseteq Q_0, \tilde{Q} \ni x} \text{esssup}_{\xi \in \tilde{Q}} |T(f\chi_{3Q_0 \setminus 3\tilde{Q}})(\xi)|,$$

which is a normal maximal operator, with an inner esssup for technical reasons: we will need this operator to be well-behaved even in its “bad set”, so we introduce an esssup so that our cubes Q_j containing a small amount of “good points” is enough to ensure $|T(f\chi_{3Q \setminus 3Q_j})|$ is bounded by the operator at the good points for any point in Q_j ¹¹, which is in turn bounded by one of the averages in the sparse form.

So, we will need our “bad set” to contain not only the “bad points” associated to $|T(f\chi_{\alpha Q})|$ (we denote this set A), but also contain the “bad points” associated to $\mathcal{M}_{Q_0, T}f(x)$ (we denote this set C), as well as contain “good points” associated to it, with small but non-zero measure (see Picture 5). We now have the question of how to find such a “bad set”, followed by the question of how to decompose it into squares. The idea here will be to apply Calderón-Zygmund decomposition to the characteristic function of the set $A \cup C$. Unlike in the case of the maximal function, this set does not decompose neatly into cubes, but we can find cubes that contain this set and have only slightly larger measure - this turns out to actually be helpful for us, since it means it captures some of the “good points” of $\mathcal{M}_{Q_0, T}f(x)$ we wanted to include.

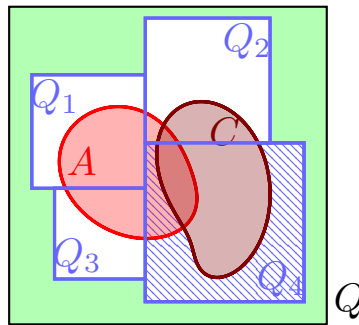


Figure 5: Visual representation of the proof structure of the iterative step of the A_2 theorem. The set A is the equivalent of the “bad set” in the maximal function, and C is the “bad set” associated to the compensating term appearing for technical reasons. Our total “bad set” is a collection of cubes with relatively small total measure containing A , C , and some “anchor points” outside of C which keep the compensating term in check for points inside the bad set.

¹¹There are some technical details here, since the esssup is taken almost everywhere, but having these “good points” have positive measure is enough to have this behave essentially as if it were a supremum.

There is some work to be done to formalize the proof, starting with working out the constants, namely the λ s in the weak inequalities and the λ in the Calderón-Zygmund decomposition. Then, we need to prove that the operator $\mathcal{M}_{Q_0,T}f(x)$ satisfies a weak bound, in order to control the size of C . Finally, we need to check that the iterative step does what we need - that it ends up covering the whole space and that the family of sets it constructs is sparse. This final part is very similar to the simpler dyadic maximal function proof.

3 The Sparse Revolution

There is a lot that can be said about the applications of Sparse Domination. In this subsection, we summarize the topics that can be read more extensively in the main text and cite some papers on the topics for further reading:

- 3.1 The main reference for the generalized pointwise Sparse Domination Result is [LO20], which in addition to the generalized theorem contains several examples in Section 5.
- 3.2 Instead of a pointwise bound, one can consider a bilinear bound (p, r) of the form $\langle Tf, g \rangle \lesssim \sum_{Q \in \mathcal{S}} |Q| \langle f \rangle_{p,Q} \langle g \rangle_{r,Q}$, which recovers L^q bounds for $p < q < r'$ as well as some weighted bounds [BFP16; Li17]. General theorems on these types of bounds can be found in [BRS24; Con+17; Bar17; Con+22] and several applications to specific operators have been proven, such as [Lac19; BC20; Bor+23; LS17; Obe19; LMR19].
- 3.3 It is possible to recover weak $(1, 1)$ bounds from $(r, 1)$ bilinear bounds (see [Con+17], appendix B). There are also several weighted bounds one can prove for endpoint cases (see [HP15; DLR16; FN19; NS24]), and many operator-specific endpoint cases have been considered, such as [KL18; Che+18; Con+24; Hu20; LTH20; BRS23].
- 3.4 It turns out that the strong maximal function can not be bounded by bi-parameter dyadic sparse forms [Bar+19] and neither can Zygmund type and Flag type maximal functions [FLW23]. There are some positive results on Sparse bounds for the bi-parameter Martingale transform via Square functions [BP17], but the results are far from sharp.

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